Sums of lognormal random variables

University of Queensland and Aarhus University Denmark

Patrick J. Laub

p.laub@[uq.edu.au|math.au.dk]

Introduction

Take a collection of n normal random variables which exhibit dependence $(X_1, \ldots, X_n) \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, and consider the sum of their exponents $S_n = \mathrm{e}^{X_1} + \cdots + \mathrm{e}^{X_n}$. This random quantity, called a sum of dependent lognormals, stubbornly refuses to answer even simple questions. For example, just calculating the $\mathbb{P}(S_n < s)$ for some $s \in \mathbb{R}_+$ is an involved numerical exercise (no closed form exists).

This variable is central to the modelling of the power of wireless signals, and is of great interest to financial risk managers. The value of a portfolio of n stocks at a future time is distributed as S_n in the Black–Scholes framework. As such, the important financial risk measure Value-at-Risk can be estimated using quantiles of S_n.

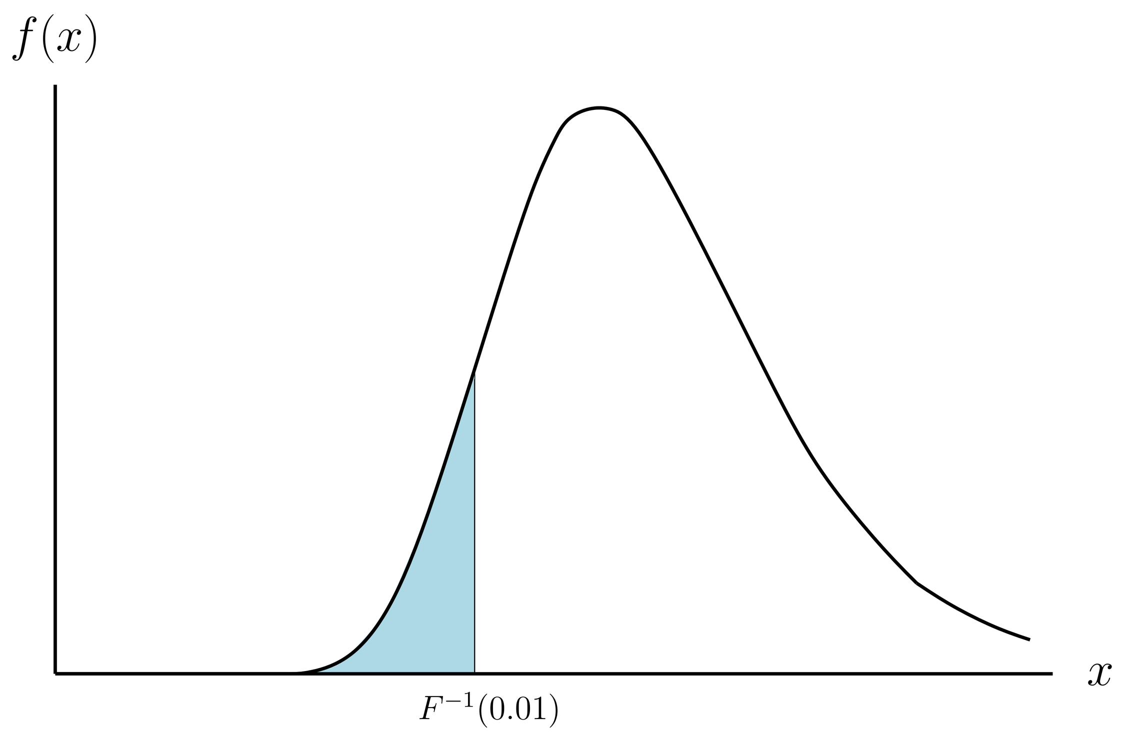

Figure 1: The 1% quantile and density function for the sum of two correlated lognormals.

We look at the Laplace transform of S_n, $\mathcal{L}(\theta) \overset{\text{def}}{=} \mathbb{E}[\mathrm{e}^{-\theta S_n}]$, which is

$$\mathcal{L}(\theta) = \frac{1}{\sqrt{(2\pi)^n \det(\boldsymbol{\Sigma})}} \int_{\mathbb{R}^n} \exp\left\{ -\theta \sum_{i=1}^{n} \mathrm{e}^{\mu_i + x_i} - \frac{1}{2} \boldsymbol{x}^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{x} \right\} \mathrm{d}\boldsymbol{x}\,.$$

The approximation

Say that $\mathcal{L}(\theta) \propto \int \exp\{-h_\theta(\boldsymbol{x})\}\,\mathrm{d}\boldsymbol{x}$ and approximate $h_\theta(\cdot)$ by a second order Taylor expansion about its minimiser $\boldsymbol{x}^*$. Then we achieve

$$\mathcal{L}(\theta) \approx \widetilde{\mathcal{L}}(\theta) \overset{\text{def}}{=} \frac{1}{\sqrt{\det(\boldsymbol{I} + \boldsymbol{\Sigma}\boldsymbol{\Lambda})}} \exp\left\{ \left(\boldsymbol{1} - \frac{1}{2}\boldsymbol{x}^*\right)^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{x}^* \right\} \qquad (1)$$

where $\boldsymbol{\Lambda} \overset{\text{def}}{=} \theta \operatorname{diag}(\mathbf{e}^{\boldsymbol{\mu} + \boldsymbol{x}^*})$. How does the error behave in $\widetilde{\mathcal{L}}(\theta)$? We found an expression for the error factor $I(\theta)$, that is, $\mathcal{L}(\theta) = \widetilde{\mathcal{L}}(\theta) I(\theta)$, and showed that $I(\theta) \to 1$ as $\theta \to \infty$. This means our approximation $\widetilde{\mathcal{L}}(\theta)$ is exact as $\theta \to \infty$. Estimating

$$I(\theta) = \sqrt{\det(\boldsymbol{I} + \boldsymbol{\Sigma}\boldsymbol{\Lambda})}\,\mathbb{E}[v(\boldsymbol{\Sigma}^{1/2} Z)] \qquad (2)$$

where $v(\boldsymbol{u}) \overset{\text{def}}{=} \exp\left\{ (\boldsymbol{x}^*)^\top \boldsymbol{\Sigma}^{-1} (\mathbf{e}^{\boldsymbol{u}} - \boldsymbol{1} - \boldsymbol{u}) \right\}$ and $Z \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{I})$ can be done using (quasi-)Monte Carlo integration.

The minimiser

The minimiser $\boldsymbol{x}^*$, found by $\theta \mathbf{e}^{\boldsymbol{\mu} + \boldsymbol{x}^*} + \boldsymbol{\Sigma}^{-1} \boldsymbol{x}^* = \boldsymbol{0}$, behaves strangely as $\theta \to \infty$. We write iterated logarithms as subscripts, i.e., $\log_2 x = \log\log x$, $\log_3 x = \log\log_2 x$, and so on.

Theorem 1. *The asymptotic form of $\boldsymbol{x}^*$ as $\theta \to \infty$ is*

$$x_i^* = \sum_{j=1}^{n} \beta_{i,j} \log_j \theta - \mu_i + c_i + \mathrm{o}(1) \qquad (3)$$

for some $\boldsymbol{\beta} = (\beta_{i,j}) \in \mathbb{R}^{n \times n}$, $\boldsymbol{c} = (c_1, \ldots, c_n)^\top \in \mathbb{R}^n$.

We also found an algorithm which determines the $\boldsymbol{\beta}$ and $\boldsymbol{c}$ constants.

Also, we can rewrite the defining equation of $\boldsymbol{x}^*$ as a system of equations involving Lambert W functions. Take $\boldsymbol{A} = \boldsymbol{\Sigma}^{-1} - \operatorname{diag}(\boldsymbol{\Sigma}^{-1})$, then

$$x_i^* = -\mathcal{W}\left(\frac{\theta \mathrm{e}^{\mu_i}}{(\boldsymbol{\Sigma}^{-1})_{i,i}} \exp\left\{ -\frac{\boldsymbol{A}_{i,\bullet}\boldsymbol{x}^*}{(\boldsymbol{\Sigma}^{-1})_{i,i}} \right\} \right) - \frac{\boldsymbol{A}_{i,\bullet}\boldsymbol{x}^*}{(\boldsymbol{\Sigma}^{-1})_{i,i}}\,.$$

Numerical Results

Below are some numerical results for estimating $\mathcal{L}(\theta)$. $\widetilde{\mathcal{L}}$ is defined by (1), $\widehat{\mathcal{L}}_{\mathrm{CMC}}$ is the crude Monte Carlo (CMC) estimator, $\widehat{\mathcal{L}}_{\mathrm{IS}}$ is $\widetilde{\mathcal{L}}$ times the CMC estimate of (2), and $\widehat{\mathcal{L}}_{\mathrm{Q}}$ is the same as $\widehat{\mathcal{L}}_{\mathrm{IS}}$ evaluated using quasi-Monte Carlo (specifically, the SOBOL sequence).

θ	100	2,500	5,000	7,500	10,000
$\widetilde{\mathcal{L}}$	-9.89 × 10^{-3}	-1.27 × 10^{-2}	-1.28 × 10^{-2}	-1.27 × 10^{-2}	-1.27 × 10^{-2}
$\widehat{\mathcal{L}}_{\mathrm{CMC}}$	1.29 × 10^{-2}	*	*	*	*
$\widehat{\mathcal{L}}_{\mathrm{IS}}$	3.36 × 10^{-4}	2.96 × 10^{-4}	2.57 × 10^{-4}	2.31 × 10^{-4}	2.11 × 10^{-4}
$\widehat{\mathcal{L}}_{\mathrm{Q}}$	-3.19 × 10^{-6}	-5.03 × 10^{-6}	-5.31 × 10^{-6}	-5.56 × 10^{-6}	-5.98 × 10^{-6}

Table 1: Relative error for various approximations of $\mathcal{L}(\theta)$. A * means the estimator failed to give a non-zero estimate.

Numerical Laplace transform algorithms have been applied to $\widehat{\mathcal{L}}_{\mathrm{Q}}$ to estimate $f(x)$. I have discovered truly remarkable inversion estimates of this form which this poster is too small to contain.

Future Directions

- The current approximation works only for $\Im(\theta) = 0$. Presumably it can be extended to the complex half-plane.
- Compare with alternative methods to approximate the distribution function $F(x)$. This includes orthogonal polynomial approximations and also fits with the log skew normal distribution.

Conclusions

- Elegant closed form approximation for $\mathcal{L}(\theta)$ given.
- Accurate Monte Carlo estimators for $I(\theta)$ have been found.
- The peculiar asymptotic form of $\boldsymbol{x}^*$ is elucidated.

References

Laub, P.J., Asmussen, S., Jensen, J.L., Rojas-Nandayapa, L. (2016) ‘Approximating the Laplace transform of the sum of dependent lognormals’. Advances in Applied Probability, 48(A), pp. 203–215.